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COMPLETE EMBEDDINGS OF LINEAR ORDERINGS AND EMBEDDINGS OF LATTICE-ORDERED GROUPS

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ABSTRACT

An infinite linearly ordered set $(S, \leqq)$ is called doubly homogeneous, if its automorphism group $\mathrm{Aut}(S, \leqq)$ acts 2-transitively on it. We study embeddings of linearly ordered sets into Dedekind-completions of doubly homogeneous chains which preserve all suprema and infima, and obtain necessary and sufficient conditions for the existence of such embeddings. As one of several consequences, for each lattice-ordered group G and each regular uncountable cardinal $\kappa \geqq |G|$ there are 2^κ non-isomorphic simple divisible lattice-ordered groups H of cardinality κ all containing G as an l-subgroup.

§1. Introduction

An infinite linearly ordered set ("chain") $(S, \leqq)$ is called doubly homogeneous, if its automorphism group, i.e. the group of all order-preserving permutations, $A(S) = \mathrm{Aut}((S, \leqq))$ acts 2-transitively on it. Chains $(S, \leqq)$ of this type and their automorphism groups $A(S)$ have been extensively studied. They have been used e.g. for the construction of infinite simple torsion-free groups (Higman [8]), or, in the theory of lattice-ordered groups (l-groups), in dealing with embeddings of arbitrary l-groups into simple divisible l-groups (Holland [9]). They also have been used for the construction of certain partially ordered sets with transitive automorphism groups ([2, 5]). Obviously, all linearly ordered fields are examples for such chains. For a variety of further results see Glass [7].

As is well-known, any linearly ordered set can be embedded into a doubly homogeneous chain (even into a linearly ordered field). However, these embeddings usually do not preserve arbitrary suprema or infima. Therefore here we study embeddings of linear orderings or their Dedekind-completions into the

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Dedekind-completion of doubly homogeneous chains which preserve all (existing) suprema and infima.

Before stating our results, let us introduce some notation. Let $(C, \leqq)$, $(D \leqq)$ be two arbitrary chains. An embedding φ of $(C \leqq)$ into $(D, \leqq)$ is called *complete*, if φ preserves all suprema and infima of subsets of C that happen to exist in $(C, \leqq)$. We denote the Dedekind-completion of the chain $(C, \leqq)$ by $(\bar{C}, \leqq)$. A non-trivial closed interval of C is a set of the form $[a, b] = \{c \in C;\ a \leqq c \leqq b\}$ where $a, b \in C$ with $a < b$. Recall that $(C, \leqq)$ is *dense* if whenever $a, b \in C$ with $a < b$, there is $c \in C$ with $a < c < b$. Now let $(C, \leqq)$ be dense and unbounded. For each point $a \in \bar{C}$ we put

$$\mathrm{cof}(a) = \min\{|A|;\ A \subseteq \bar{C},\ a \notin A,\ a = \sup A\}, \qquad \text{the } \textit{cofinality} \text{ of } a,$$

and

$$\mathrm{coi}(a) = \min\{|A|;\ A \subseteq \bar{C},\ a \notin A,\ a = \inf A\}, \qquad \text{the } \textit{coinitiality} \text{ of } a.$$

If $\mathrm{cof}(a) = \mathrm{coi}(a)$, this is called the *coterminality* of a. The chain $(C, \leqq)$ (or $(\bar{C}, \leqq)$) has *countable coterminality*, if it contains a countable subset which is unbounded both above and below in C. Using a result of Droste and Shelah [6], we will show:

THEOREM 1. *Let $(C, \leqq)$ be an arbitrary chain. The following are equivalent*:

(1) *No non-trivial closed interval of C is dense.*

(2) *There exists a doubly homogeneous chain $(S, \leqq)$ and a complete embedding φ of $(\bar{C}, \leqq)$ into $(\bar{S}, \leqq)$ such that $\varphi(\bar{C}) \subseteq \bar{S} \setminus S$.*

Moreover, if one of these conditions is saisfied and $\kappa \geqq |C|$ is any infinite cardinal, then the chain $(S, \leqq)$ and the embedding φ in condition (2) *may be chosen such that, in addition, S has cardinality κ, each point $s \in S$ has countable coterminality, and whenever $a, b \in C$ with $a < b$ and no $c \in C$ satisfies $a < c < b$, then the interval $(\varphi(a), \varphi(b))$ in $\bar{S}$ has countable coterminality.*

Note that the embedding result of Theorem 1 applies in particular to the class of all scattered linear orderings, since any such chain satisfies the assumption (1) of Theorem 1. A special case of Theorem 1 has already been used, in a natural way, when constructing all normal subgroup lattices of the automorphism groups $A(S)$ (S a doubly homogeneous chain), cf. [6; §4]. Other special cases of this result have been used in [3, 5].

Let again $(C, \leqq)$ be an arbitrary chain. We will call a pair (A, B) of non-empty disjoint subsets A, B of C a *Dedekind-hole* in C if $C = A \dot{\cup} B$, $a < b$ for all $a \in A$, $b \in B$, and A has a greatest element iff B has a smallest element. The set of all Dedekind-holes of C is said to be dense in C if whenever $a, b \in C$ with $a < b$, there exists a Dedekind-hole (A, B) of C with $a \in A$, $b \in B$. As a consequence of Theorem 1 we will derive

COROLLARY 1. *Let* $(C, \leqq)$ *be an arbitrary chain. The following are equivalent*:

(1) *The set of Dedekind-holes of* C *is dense in* C.

(2) *There exists a doubly homogeneous chain* $(S, \leqq)$ *and a complete embedding* φ *of* $(C, \leqq)$ *into* $(\bar{S}, \leqq)$ *such that* $\varphi(C) \subseteq \bar{S} \setminus S$.

Moreover, if one of these conditions is satified, the chain $(S, \leqq)$ *and the embedding* φ *in condition* (2) *may be chosen such that, in addition,* S *has cardinality* $|\bar{C}|$, *each point* $s \in S$ *has countable coterminality, and whenever* (A, B) *is a Dedekind-hole of* C, *then* $\sup \varphi(A) < \inf \varphi(B)$ *in* $(\bar{S}, \leqq)$ *and the interval* $(\sup \varphi(A), \inf \varphi(B))$ *in* $\bar{S}$ *has countable coterminality.*

Observe here that the class of all linear orderings with a dense set of Dedekind-holes includes the chains satisfying condition (1) of Theorem 1, but also dense chains like the rationals and the irrationals, more generally all sets C for which $\bar{C} \setminus C$ is dense in $\bar{C}$. As a further consequence of Theorem 1 and a result of Solovay [18], we will show:

THEOREM 2. *For each regular uncountable cardinal* κ *and all regular cardinals* $\mu, \nu \leqq \kappa$, *there exist up to isomorphism precisely* 2^{κ} *doubly homogeneous chains* $(S, \leqq)$ *of cardinality* κ *with pairwise non-isomorphic and non-antiisomorphic Dedekind-completions such that each point* $s \in S$ *has cofinality* μ *and coinitiality* ν.

We will also obtain an answer to a question of van Benthem [19] which arose in a study of tense logic and time.

For any chain $(S, \leqq)$, $A(S)$ is a lattice-ordered group (l-group) by the pointwise ordering of functions (for $f, g \in A(S)$, put $f \leqq g$ iff $f(s) \leqq g(s)$ for all $s \in S$). If S is doubly homogeneous, then $B(S)$, the l-subgroup (subgroup sublattice) of $A(S)$ comprising all automorphisms of S with bounded support, is algebraically simple and divisible (Higman [8], Holland [9]). If G, H are l-groups and $\psi: G \to H$ is a group-embedding which preserves the lattice operations ($\vee$ and $\wedge$), then ψ is an l-embedding. Now let $G \subseteq A(S)$, $H \subseteq A(T)$ be l-subgroups for some chains S, T, and let $\varphi: S \to T$ be an embedding and $\psi: G \to H$ an l-embedding. Then (ψ, φ) is called an l-embedding of (G, S) into (H, T), if $\varphi \circ g = \psi(g) \circ \varphi$ for all $g \in G$. We will show:

THEOREM 3. *Let* $(C, \leqq)$ *be an arbitrary chain and* $\kappa \geqq |C|$ *a regular uncount-*

able cardinal. Then there are 2^κ doubly homogeneous chains $(S, \leqq)$ of cardinality κ with pairwise non-isomorphic and non-antiisomorphic Dedekind-completions such that $(A(C), C)$ can be l-embedded into $(A(S), S)$, or, in fact, even into $(B(S), S)$.

Note that this result contains, in particular, our starting-point that any chain can be embedded into a doubly homogeneous chain as a special case again. For many of the sets $(S, \leqq)$ constructed here, the automorphism groups $A(S)$ are elementarily inequivalent as groups. Using Holland's theorem [9] that any l-group G can be l-embedded into $A(C)$ for some chain C and a result of McClearly [15], we obtain as a consequence of Theorem 3:

COROLLARY 2. *Let G be an arbitrary l-group and $\kappa \geqq |G|$ a regular uncountable cardinal. Then there exist 2^κ simple divisible l-groups H of cardinality κ which are pairwise non-isomorphic as groups and all contain G as an l-subgroup.*

Theorem 3 and Corollary 2 sharpen results of Holland [9] and Weinberg [20]. We will also consider embeddings of arbitrary l-groups into l-groups all of whose group automorphisms are inner, and we show how to obtain similar results for singular cardinals.

The proofs of Theorems 1, 2, 3 are contained in Sections 2, 3, 4, respectively. For the basic background on our topic we refer the reader to Glass [7].

§2. Complete embeddings of linear orderings

In this section we prove Theorem 1 and Corollary 1 and derive a few related results on complete embeddings. First let us show that embeddings of linear orderings into (doubly) homogeneous chains which preserve all suprema (or all infima) are rare. An infinite chain $(S, \leqq)$ is called homogeneous if $A(S)$ acts transitively on it.

PROPOSITION 2.1. *Let $(C, \leqq)$ be a linear ordering for which there exists an embedding of the ordinal $\omega_1 + 1$ into C which preserves all suprema. Let $(S, \leqq)$ be any homogeneous chain. Then there is no embedding of $(C, \leqq)$ into $(S, \leqq)$ which preserves all suprema.*

PROOF. Assume there is an embedding φ of $\omega_1 + 1$ into $(S, \leqq)$ which preserves all suprema. Let $a \in \omega_1 + 1$ be a countable limit-ordinal and b the maximal element of $\omega_1 + 1$. Then $\mathrm{cof}(\varphi(a)) = \aleph_0$ and $\mathrm{cof}(\varphi(b)) = \aleph_1$ in S, thus there is no automorphism of S mapping $\varphi(a)$ onto $\varphi(b)$. Hence $(S, \leqq)$ is not homogeneous, and the result follows.

Now we turn to the proof of Theorem 1. We first need a few preparations. If $(C,\leqq)$ is a chain, $Z \subseteq \bar{C}$, and $a, b \in \bar{C}$ with $a < b$, we let $(a,b)_Z = \{z \in Z; a < z < b\}$; also $(-\infty, a)_Z = \{z \in Z; z < a\}$. The intervals $[a,b]_Z$ and $(a,\infty)_Z$ are defined similarly, and the index Z is omitted if this unambiguous.

Definition 2.2 ([6]). Let κ be a cardinal. A chain $(C,\leqq)$ is called a good κ-set, if the following conditions are satisfied:

(1) $|C| = \kappa$, and $(C,\leqq)$ is dense and unbounded.

(2) Each $c \in C$ has countable coterminality.

(3) Whenever $x, y \in C$ with $x < y$, there is a set $A \subseteq [x,y]_{\bar{C}} \setminus C$ such that $|A| = \kappa$ and each $a \in A$ has countable coterminality.

Obviously, any good $\aleph_0$-set is isomorphic to $\mathbf{Q}$, the set of all rationals. Now we deal with the existence problem of good κ-sets for arbitrary cardinals κ:

Lemma 2.3 ([6; Lemma 4.2]). *Let κ be a cardinal. Then there exists a good κ-set $(C,\leqq)$ of countable coterminality.*

If $(C,\leqq)$ is a chain and $A, B \subseteq C$, we write $A < B$ $(A \leqq B)$ to denote that $a < b$ $(a \leqq b)$ for all $a \in A$, $b \in B$. Also let $x < A$ $(A < x$, $A \leqq x$ etc.) abbreviate $\{x\} < A$ $(A < \{x\}$, $A \leqq \{x\})$, respectively. Now let $(S_1,\leqq)$, $(S_2,\leqq)$ be two chains with $(S_1,\leqq) \subseteq (S_2,\leqq)$, i.e. $S_1 \subseteq S_2$ and the order of S_2 extends the order of S_1. Then S_1 is dense in S_2 if whenever $a, b \in S_2$ with $a < b$, there is $x \in S_1$ with $a < x < b$. Hence S_1 is dense iff S_1 is dense in itself. Let $a \in \bar{S}_1$. We put

$$\mathrm{Ded}(a, S_1, S_2) = \{z \in S_2; \{s \in S_1; s < a\} < z < \{s \in S_1; a < s\}\}.$$

For later purposes note that we have $\mathrm{Ded}(a, S_1, S_2) = \varnothing$ iff $a \notin S_2$ and for any $z \in S_2$ with $z < \{s \in S_1; a < s\}$ $(\{s \in S_1; s < a\} < z)$ there is $x \in S_1$ with $z \leqq x < a$ $(a < x \leqq z)$, respectively. Now we give the

Proof of Theorem 1. (2)$\rightarrow$(1): Let $a, b \in C$ with $a < b$. Since S is doubly homogeneous, it is dense, and we can choose $s \in S$ with $\varphi(a) < s < \varphi(b)$. Let $A = \{z \in \bar{C}; \varphi(z) \leqq s\}$ and $B = \{z \in \bar{C}; s \leqq \varphi(z)\}$, and put $c = \sup A$, $d = \inf B$ in $(\bar{C},\leqq)$. Then $\varphi(c) = \sup \varphi(A)$, $\varphi(d) = \inf \varphi(B)$ in $(\bar{S},\leqq)$ and $\varphi(c) < s < \varphi(d)$ by $s \notin \varphi(\bar{C})$. Thus $c < d$, and no $z \in \bar{C}$ satisfies $c < z < d$. Hence $c, d \in C$, showing that the interval $[a,b]_C$ is not dense.

(1)$\rightarrow$(2): We show that we can find a doubly homogeneous chain $(S,\leqq)$ satisfying also the additional properties stated in the theorem. So let $\kappa \geqq |C|$ be any infinite cardinal. We may assume that C has no greatest and no smallest element (otherwise "enlarge" C correspondingly). First let us assume that κ is regular.

Let P be the set of all pairs $(a \mid b)$ such that $a, b \in C$, $a < b$, and no $c \in C$ satisfies $a < c < b$. For each pair $(a \mid b) \in P$ choose by Lemma 2.3 a good κ-set $L_{(a|b)}$ with countable coterminality, and let

$$S_0 = \dot{\bigcup_{(a|b) \in P}} L_{(a|b)}.$$

Next we define a linear order on $\bar{C} \dot{\cup} S_0$ in the natural way such that it extends the given orders of $\bar{C}$ and of each $L_{(a|b)}$ and $a < L_{(a|b)} < b$ in $(\bar{C} \dot{\cup} S_0, \leqq)$ for each $(a \mid b) \in P$. Note that whenever $c, d \in C$ with $c < d$, by assumption (1) there are $a, b \in C$ such that $c \leqq a < b \leqq d$ and $(a \mid b) \in P$. This shows first $P \neq \emptyset$, hence $|S_0| = \kappa$ and $(S_0, \leqq)$ is a good κ-set. Secondly, we also obtain that whenever $c, d \in \bar{C}$ with $c < d$, there is $s \in S_0$ with $c < s < d$. Thus S_0 is dense in $S_0 \dot{\cup} \bar{C}$, and hence we may regard $\bar{C}$ as a subset of $\overline{S_0} \setminus S_0$. Thirdly, $\bar{C}$ is unbounded above and below in $\overline{S_0}$.

Now if $\kappa = \aleph_0$, we put $S = S_0$ and let φ be the identity mapping. Since then S is isomorphic to $\mathbf{Q}$, $A(S)$ is doubly homogeneous. As φ is complete (as indicated in detail below), the result follows. Therefore let now κ be regular and uncountable. We follow the construction in the proof of Theorem 2.11, parts (I)–(III), of Droste and Shelah [6] almost literally, with $T_0 = \bar{C}$, $\lambda = \kappa$, $\Omega_i = S_i$. Hence there are chains $(S_i, \leqq)$ $(i \in \kappa)$ such that $|S_i| = \kappa$ and $(S_0, \leqq) \subseteq (S_i, \leqq) \subseteq (S_j, \leqq)$ for all $i, j \in \kappa$ with $i < j$, $\mathrm{Ded}(a, S_0, S_i) = \emptyset$ for each $a \in \bar{C}$ (cf. [6; conditions (4.7) (I), (III)]), and if $(S, \leqq)$ is defined in the natural way such that $S = \bigcup_{i \in \kappa} S_i$ and the order of S extends the order of each S_i $(i \in \kappa)$, then $(S, \leqq)$ is a doubly homogeneous chain and each point $s \in S$ has countable coterminality. It follows that $|S| = \kappa$ and also $\mathrm{Ded}(a, S_0, S) = \emptyset$ for each $a \in \bar{C}$. In particular, this shows $\bar{C} \cap S = \emptyset$ and if $a \in \bar{C}$, $A \subseteq S_0$, and $a = \sup A$ $(a = \inf A)$ in $S_0 \dot{\cup} \bar{C}$, then also $a = \sup A$ $(a = \inf A)$ in $S \cup \bar{C}$. Hence $\bar{C}$ is also a subset of $\bar{S}$, thus $\bar{C} \subseteq \bar{S} \setminus S$, and we may define φ to be the identity mapping. Clearly φ preserves all suprema and infima of subsets of $\bar{C}$, since if e.g. $a = \sup A$ in $\bar{C}$ for some $a \in \bar{C}$ and $A \subseteq \bar{C}$, then also $a = \sup B$ in S_0 for $B = \{s \in S_0;\ s < a\}$ by $\bar{C} \subseteq \overline{S_0}$. Hence also $a = \sup B$ in $\bar{S}$; now the fact that A is unbounded above in $A \cup B$ implies that $a = \sup A$ in $\bar{S}$. Also, whenever $(a \mid b) \in P$, the set $L_{(a|b)}$ is unbounded above and below in $\{s \in \bar{S}\colon a < s < b\}$. Since $L_{(a|b)}$ has countable coterminality by construction, it follows that $(a, b)_S$ also has countable coterminality. Hence we have shown the result in this case.

If κ is a singular cardinal, we proceed in almost precisely the same way. We only have to take into [6; Definition 4.9] the additional requirement that if λ is a singular cardinal, then also $|\mathrm{Jump}(\Omega_i, \Omega_k)| \leqq \aleph_k$ whenever $i < k < j$ (in the notation present there). Then [6; Lemma 4.10] holds also for singular cardinals λ. Observe that in [6; Lemma 4.8] we have also

$$|\mathrm{Jump}(\Omega_1,\Omega_3)| \leqq |\mathrm{Jump}(\Omega_1,\Omega_2)| + |\mathrm{Jump}(\Omega_2,\Omega_3)|,$$

as shown there in the proof. Now our proof goes through as before.

The following result, an immediate consequence of the argument just given, was also noted by S. Shelah.

COROLLARY 2.4. *Theorem* 2.11 *of Droste and Shelah* [6] *holds also for singular cardinals* λ.

We just note that now the use of Theorem 1 allows us to considerably simplify the argument for Theorem 2.11 of Droste and Shelah [6] (note that the set T_0 constructed in the proof of [6; Theorem 2.11] satisfies assumption (1) of Theorem 1, apply Theorem 1 to obtain the doubly homogeneous chain $(\Omega, \leqq)$, and continue with part (IV) of the proof of Theorem 2.11).

The existence result of Theorem 1 will be sharpened in Theorem 3.2. We note that the chain S constructed in the above proof has some additional interesting point character properties. For this, we need some more notation. Let $(D, \leqq)$ be a Dedekind-complete chain, not necessarily dense. Let $a \in D$. If a has no immediate predecessor (successor), we define the cofinality (coinitiality) of a as before; otherwise we put $\mathrm{cof}(a) = \aleph_0$ ($\mathrm{coi}(a) = \aleph_0$), respectively. Then the pair $(\mathrm{cof}(a), \mathrm{coi}(a))$ is called the *character* of a. Let $\mathrm{Char}(D)$ denote the set of all characters of elements $a \in D$. We wish to determine $\mathrm{Char}(\bar{S})$. It is easy to check that the good κ-set L constructed in [6; Lemma 4.2] (and used in the above proof) satisfies

$$\mathrm{Char}(\bar{L}) = \{(\mu, \aleph_0);\ \aleph_0 \leqq \mu \leqq \kappa,\ \mu \text{ regular}\}.$$

Clearly $\mathrm{Char}(\bar{L}) \cup \mathrm{Char}(\bar{C}) \subseteq \mathrm{Char}(\bar{S})$. We may even obtain equality here in the following way. Whenever $j < \kappa$ is a countable limit-ordinal, $I \subseteq \kappa$ with $j = \sup I$, and $a_i, b_i \in S_{i+1} \setminus S_i$ with $a_i < a_{i+1} < b_{i+1} < b_i$ for each $i \in I$, then in $S_j = \bigcup_{i<j} S_i$ we have

$$\sup\{a_i\,;\, i \in I\} = \inf\{b_i\,;\, i \in I\} =: x \in \bar{S}_j \setminus S_j$$

and x has coterminality $\aleph_0$ in $\bar{S}_j$. In order to get $\mathrm{Ded}(x, S_j, S) = \varnothing$ and hence $\mathrm{cof}(x) = \mathrm{coi}(x) = \aleph_0$ in $\bar{S}$, we let T_j, the set of all "forbidden points" of $\bar{S}_j \setminus S_j$, contain $\bigcup_{i<j} T_i$ and all elements $x \in \bar{S}_j \setminus S_j$ of the form described above (cf. [6; pp. 251, 255]). Continuing with the construction of S as before, we obtain:

COROLLARY 2.5. *Let* $(C, \leqq)$ *be a chain in which no non-trivial closed interval is dense, and let* $\kappa \geqq |C|$ *be any infinite cardinal. Then there exists a doubly homogeneous chain* $(S, \leqq)$ *of cardinality* κ *satisfying the assertion of Theorem* 1, *and, in addition,*

$$\mathrm{Char}(\bar{S}) = \{(\mu, \aleph_0);\ \mu \leqq \kappa,\ \mu \text{ regular}\} \cup \mathrm{Char}(\bar{C}).$$

Next we use Theorem 1 to give the

PROOF OF COROLLARY 1. (1)→(2): We show that we can find a doubly homogeneous chain $(S, \leqq)$ satisfying also the additional properties stated in the corollary. Let H be the set of all Dedekind-holes (A, B) of C for which A has no greatest and thus B no smallest element. For each $(A, B) \in H$, choose two elements a_A, a_B. Let

$$D = C \,\dot{\cup} \dot{\bigcup_{(A,B)\in H}} \{a_A, a_B\},$$

and define a linear order $\leqq$ on D in the natural way such that it extends the order of C and $A < a_A < a_B < B$ in $(D, \leqq)$ for each $(A, B) \in H$. Then $|D| = |\bar{C}|$, D is Dedekind-complete, the identity mapping which embeds $(C, \leqq)$ into $(D, \leqq)$ is complete, and the assumption (1) shows that no non-trivial closed interval of D is dense. Hence we may apply Theorem 1 on $(D, \leqq)$ which immediately implies the result.

(2)→(1): Let $a, b \in C$ with $a < b$. Since S is dense, there is $s \in S$ with $\varphi(a) < s < \varphi(b)$. Let $A = \{c \in C; \varphi(c) < s\}$ and $B = \{c \in C; s < \varphi(c)\}$. Then $a \in A$, $b \in B$, $C = A \,\dot{\cup}\, B$ as $s \notin \varphi(C)$, and $A < B$. It remains to show that A has a greatest element iff B has a smallest element. So assume without loss of generality that A has a greatest element $c \in A$. If B has no smallest element, we would obtain $c = \inf B$ in $(C, \leqq)$ and thus $\varphi(c) = \inf \varphi(B)$ in $(\bar{S}, \leqq)$ in contradiction to $\varphi(c) < s < \varphi(B)$. Hence B has a smallest element, and the result follows.

In view of Theorem 1 we now show that not every chain can be completely embedded into the Dedekind-completion of a doubly homogeneous chain.

PROPOSITION 2.6. *Let* $(C, \leqq)$ *be a dense unbounded chain such that* $(\bar{C}, \leqq)$ *is not the Dedekind-completion of a doubly homogeneous chain. Let* $(S, \leqq)$ *be any doubly homogeneous chain. Then there is no complete embedding* φ *of* $(\bar{C}, \leqq)$ *into* $(\bar{S}, \leqq)$.

PROOF. Suppose such a complete embedding φ exists. Then, as shown in the proof of Theorem 1, $\varphi(\bar{C})$ is a convex subset of $\bar{S}$, i.e. whenever $a, b \in \bar{C}$ and

$s \in \bar{S}$ with $\varphi(a) < s < \varphi(b)$, then $s \in \varphi(\bar{C})$. Hence $\varphi(\bar{C})$ is the Dedekind-completion of $\varphi(\bar{C}) \cap S$, and the chain $\varphi(\bar{C}) \cap S$ is doubly homogeneous, a contradiction.

However, Propositions 2.1 and 2.6 are contrasted by the following consequence of Theorem 1.

COROLLARY 2.7. *Let $(C, \leqq)$ be an arbitrary chain and $\kappa \geqq |C|$ any infinite cardinal. Then there exists a doubly homogeneous chain $(S, \leqq)$ of cardinality κ and an embedding φ of $(\bar{C}, \leqq)$ into $(\bar{S}, \leqq)$ such that φ preserves all suprema of subsets of $\bar{C}$.*

PROOF. Let $D = C \times \{1,2\}$ be ordered lexicographically, i.e. $(c,i) < (c',j)$ iff either $c < c'$ or $c = c'$ and $i < j$ $(c, c' \in C, i, j \in \{1,2\})$. Then $\bar{D} = D \dot{\cup} (\bar{C} \setminus C)$ in the natural way such that $(c,i) < \bar{c}$ iff $c < \bar{c}$ $((c,i) \in D,\ \bar{c} \in \bar{C} \setminus C)$. Clearly the mapping $\pi: \bar{C} \to \bar{D}$, defined by $\pi(c) = (c,1)$ if $c \in C$, and $\pi(c) = c$ if $c \in \bar{C} \setminus C$, is an embedding and preserves all suprema of subsets of $\bar{C}$. Since $|D| \leqq \kappa$ and no non-trivial closed interval of $(D, \leqq)$ is dense, now Theorem 1 implies the result.

§3. Constructions of non-isomorphic doubly homogeneous chains

In this section we use Theorem 1 and a result of Solovay [18] to construct "many" doubly homogeneous chains of a prescribed cardinality with non-isomorphic Dedekind-completions and with several additional properties. In particular, we will prove Theorem 2.

First let us sharpen the main result of Theorem 1. As usual, we identify cardinals with the least ordinals of their cardinality. Let $(S, \leqq)$ be a chain with no greatest element. Let us call a subset $A \subseteq S$ a *club* in S, if it is closed and unbounded above in S, and *stationary* in S, if it intersects each club of S. The cardinal

$$\mathrm{cof}(S) = \min\{|A|;\ A \subseteq S \text{ is unbounded above in } S\}$$

is called the *cofinality* of S. Dually, if S is unbounded below, the cardinal

$$\mathrm{coi}(S) = \min\{|A|;\ A \subseteq S \text{ is unbounded below in } S\}$$

is called the *coinitiality* of S. If $\mathrm{cof}(S) = \mathrm{coi}(S)$, this is called the *coterminality* of S. The proof of the following lemma is straightforward.

LEMMA 3.1 (cf. [1; 3.16(c)]). *Let S be an unbounded chain with uncountable cofinality and $A, B \subseteq \bar{S}$ two clubs in $\bar{S}$. Then $A \cap B$ is also a club in $\bar{S}$. If*

$\psi: A \to B$ is an isomorphism, there exists a club $C \subseteq A \cap B$ on which ψ acts like the identity.

Now we show:

THEOREM 3.2. *Let $(C, \leqq)$ be an arbitrary chain in which no non-trivial closed interval is dense, λ any regular uncountable cardinal, and $\kappa \geqq \lambda + |C|$. Then there exist (at least) 2^λ doubly homogeneous chains $(S, \leqq)$ of cardinality κ and cofinality λ with pairwise non-isomorphic and non-antiisomorphic Dedekind-completions such that for each of these chains $(S, \leqq)$, there exists a complete embedding φ of $(\bar{C}, \leqq)$ into $(\bar{S}, \leqq)$ with the following properties*:

(i) $\varphi(\bar{C}) \subseteq \bar{S} \setminus S$;

(ii) *whenever $a, b \in C$ with $a < b$ and no $c \in C$ satisfies $a < c < b$, then the interval $(\varphi(a), \varphi(b))$ in $\bar{S}$ has countable coterminality*;

(iii) *each point $s \in S$ has countable coterminality.*

PROOF. We may assume that $(C, \leqq)$ contains a copy of the ordinal κ; otherwise enlarge $(C, \leqq)$ correspondingly. Let $(T, \leqq)$ be a well-ordered chain isomorphic to λ. For each element $a \in T$, choose a chain $(L_a, \leqq)$ which is antiisomorphic to ω_1. For each subset $Z \subseteq T$ let

$$C_Z = C \dot{\cup} \left(T \dot{\cup} \dot{\bigcup_{a \in Z}} L_a \right).$$

Next we define a linear order $\leqq$ on C_Z such that it extends the orders of C, T, and of each L_a $(a \in Z)$, $C < (T \dot{\cup} \bigcup_{a \in Z} L_a)$, and $a < L_a < \min\{t \in T;\, a < t\}$ for each $a \in Z$. Then $|C_Z| = \kappa$, and clearly no interval of C_Z is dense. Hence by Theorem 1 there is a doubly homogeneous chain $(S_Z, \leqq)$ of cardinality κ such that (without loss of generality) $\bar{C}_Z \subseteq \bar{S}_Z \setminus S_Z$, the identity mapping from $\bar{C}_Z$ into $\bar{S}_Z$ is complete, each point $s \in S_Z$ has countable coterminality, and whenever $a, b \in C_Z$ with $a < b$ and no $c \in C_Z$ satisfies $a < c < b$, then the interval (a, b) in $\bar{S}_Z$ has countable coterminality. Note that hence in $\bar{S}_Z$ we have $\mathrm{coi}(a) = \aleph_1$ for each $a \in Z$, and $\mathrm{coi}(a) = \aleph_0$ for each $a \in T \setminus Z$. We may assume that C_Z is unbounded above in $\bar{S}_Z$; otherwise let $s = \sup C_Z \in \bar{S}_Z$ and consider $\{x \in S_Z;\, x < s\}$ instead of S_Z. In particular, S_Z has cofinality λ. We can also assume that S_Z has countable coinitiality; otherwise choose $s \in \bar{S}_Z$ with $s < C_Z$ and $\mathrm{coi}(s) = \aleph_0$ and consider $\{x \in S_Z;\, s < x\}$ instead of S_Z. If follows that the chains S_Y, S_Z $(Y, Z \subseteq T)$ are never antiisomorphic. Note that since C_Z contains a copy of κ, this procedure does not change the cardinality of S_Z. In fact, since this copy of κ is bounded in S_Z, a copy of κ is contained in each non-trivial interval of $\bar{S}_Z$.

By a well-known result of Solovay [18], we can split $T = \dot{\bigcup}_{i \in \lambda} A_i$ such that each A_i is a stationary subset of T ($i \in \lambda$). For each $A \subseteq \lambda$, let $A^* = \bigcup_{i \in A} A_i \subseteq T$. We claim that the chains S_{A^*} ($A \subseteq \lambda$) satisfy the requirements of the theorem. For this, it only remains to show that whenever $A, B \subseteq \lambda$ with $A \neq B$, then $\overline{S_{A^*}}$ and $\overline{S_{B^*}}$ are not isomorphic.

Indeed, suppose $\psi: \overline{S_{A^*}} \to \overline{S_{B^*}}$ were an isomorphism. Since T and $\psi(T)$ are subsets of $\overline{S_{B^*}}$ which are closed and unbounded above in $\overline{S_{B^*}}$, by Lemma 3.1 there is a club $D \subseteq T \cap \psi(T)$ in T on which ψ acts like the identity. Since $A \neq B$, we may assume that there is some $i \in A \setminus B$. Then $A_i \subseteq A^* \subseteq T$, and since A_i is stationary in T, there exists $a \in A_i \cap D \subseteq T \subseteq \overline{S_{A^*}}$. Consequently, a has coinitiality $\aleph_1$ in $\overline{S_{A^*}}$ and, by $a = \psi(a)$, also in $\overline{S_{B^*}}$. But A_i is disjoint to B^*, hence $a \in T \setminus B^* \subseteq \overline{S_{B^*}}$ must have countable coinitiality in $\overline{S_{B^*}}$, which establishes our contradiction.

If $(T, \leqq)$ is a chain and $x \in T$, we call $(x, \infty)_T = \{t \in T;\ x < t\}$ a *final segment* of $(T, \leqq)$. As a consequence of the proof of Theorem 3.2 we note that the constructed doubly homogeneous chains $(S, \leqq)$ have the additional property that even the final segments of their Dedekind-completions are pairwise non-isomorphic and that each non-trivial interval of $(\bar{S}, \leqq)$ contains a copy of κ. The following consequence of Theorem 3.2 will be applied in [5] for the construction of certain infinite partially ordered sets with transitive automorphism groups.

COROLLARY 3.3. *Let λ be a regular uncountable cardinal and $\kappa \geqq \lambda$. Then there are (at least) 2^λ doubly homogeneous chains $(S, \leqq)$ of cardinality κ and cofinality λ with pairwise non-isomorphic and non-antiisomorphic Dedekind-completions such that for all pairs of regular cardinals $\mu, \nu \leqq \kappa$, there exists a point $s \in \bar{S} \setminus S$ with cofinality μ and coinitiality ν. Moreover, these chains $(S, \leqq)$ can be chosen such that even the final segments of their Dedekind-completions are pairwise non-isomorphic, and such that each non-trivial interval of each of the chains $(\bar{S}, \leqq)$ contains a copy of the ordinal κ.*

PROOF. Let P be the set of all pairs (μ, ν) such that $\mu, \nu \leqq \kappa$ are regular cardinals. Well-order P. For each $p = (\mu, \nu) \in P$, choose a chain A_p isomorphic to μ, a chain B_p antiisomorphic to ν, and a point z_p not belonging to A_p or B_p. Let $C_p = A_p \dot{\cup} \{z_p\} \dot{\cup} B_p$ and define a linear order $\leqq$ on C_p such that it extends the orders of A_p and B_p and $A_p < \{z_p\} < B_p$. Let $(C, \leqq)$ be the ordinal sum of the linear orders $(C_p, \leqq)$ ($p \in P$); hence $C = \bigcup_{p \in P} C_p$ and $p, q \in P$, $p < q$ imply $C_p < C_q$ in $(C, \leqq)$. Now apply Theorem 3.2 to obtain 2^λ doubly homogeneous chains $(S, \leqq)$ of cardinality κ and cofinality λ with pairwise non-isomorphic and non-antiisomorphic Dedekind-completions, and complete embeddings φ of $(\bar{C}, \leqq)$ into $(\bar{S}, \leqq)$. Then for each $p = (\mu, \nu) \in P$, the element $\varphi(z_p) \in \bar{S} \setminus S$ has cofinality μ and coinitiality ν. The final statement of the corollary is immediate by the remark following Theorem 3.2.

In applications of Theorem 3.2 and Corollary 3.3 (as, for instance, for our proof of Theorem 2), of course the most important cases are the ones where either $\kappa = \lambda$ or else κ is a singular cardinal. Now we turn to the proof of Theorem 2. Let $(S, \leqq)$ be any chain. Each automorphism f of $(S, \leqq)$ extends uniquely to an automorphism of $(\bar{S}, \leqq)$ which we will also denote by f. If $z \in \bar{S}$, the set $\{f(z);\ f \in A(S)\}$ is called the $A(S)$-orbit of z in $\bar{S}$.

LEMMA 3.4 (McCleary [14; p. 418]). *Let $(S, \leqq)$ be a doubly homogeneous chain, $z \in \bar{S}$, and U the $A(S)$-orbit of z in $\bar{S}$. Then U is a doubly homogeneous chain, U is dense in $\bar{S}$, and*

$$\mathrm{cof}(x) = \mathrm{cof}(z), \quad \mathrm{coi}(x) = \mathrm{coi}(z) \qquad \textit{for all } x \in U.$$

We will also need the following Löwenheim–Skolem-type result for doubly homogeneous chains.

LEMMA 3.5. *Let C be a doubly homogeneous chain, κ a cardinal, and $A \subseteq C$ such that $|A| \leqq \kappa \leqq |C|$. Then there exists a doubly homogeneous chain B with $A \subseteq B \subseteq C$ and $|B| = \kappa$.*

PROOF. By assumption, there exists a function f: $C^5 \rightarrow C$ such that for all $a, b, c, d \in C$ with $a < b$ and $c < d$, the function $f(a, b, c, d, \cdot)$ maps the interval $[a, b]$ isomorphically onto $[c, d]$. By the downward Löwenheim–Skolem theorem the model $\mathscr{C} = \langle C, <, f \rangle$ has an elementary submodel $\mathfrak{B} < \mathscr{C}$ whose domain B satisfies $A \subseteq B \subseteq C$ and $|B| = \kappa$. The result follows.

Next we give the

PROOF OF THEOREM 2. Let κ be any regular uncountable cardinal. Clearly there are at most 2^κ non-isomorphic chains of cardinality κ. To prove the existence part, by Corollary 3.3 we can choose 2^κ doubly homogeneous chains $(S_i, \leqq)$ $(i < 2^\kappa)$ of cardinality and cofinality κ with pairwise non-isomorphic and non-antiisomorphic Dedekind-completions such that there exists a point $s_i \in \bar{S}_i \setminus S_i$ with cofinality μ and coinitiality ν, and each non-trivial interval of $(\bar{S}_i, \leqq)$ contains a copy of κ. Now, for each $i < 2^\kappa$, choose such a point s_i and let U_i be the $A(S_i)$-orbit of s_i in $\bar{S}_i$. By Lemma 3.4, U_i is doubly homogeneous, U_i is dense in S_i, and $\mathrm{cof}(x) = \mu$, $\mathrm{coi}(x) = \nu$ for all $x \in U_i$. For all $a, b \in S_i$ with $a < b$ choose an element $c \in U_i$ with $a < c < b$, and let D_i be the set containing precisely all these elements $c = c(a, b)$. Thus $D_i \subseteq U_i$, and D_i is dense in S_i. Therefore $|D_i| = \kappa$, since κ can be embedded into S_i. By Lemma 3.5, there is a doubly homogeneous chain T_i of cardinality κ such that $D_i \subseteq T_i \subseteq U_i$. Thus T_i is also dense in S_i, showing that $T_i = S_i$ and $\mathrm{cof}(x) = \mu$, $\mathrm{coi}(x) = \nu$ for all $x \in T_i$. Consequently, the chains T_i $(i < 2^\kappa)$ satisfy the requirements of the theorem.

For algebraic properties of the automorphism groups $A(S)$ constructed here, see section 4. We just note that if κ is a singular cardinal, by the same argument there are at least $2^{<\kappa} = \Sigma_{\lambda<\kappa} 2^\lambda$ doubly homogeneous chains $(S, \leqq)$ of cardinality κ satisfying the requirements of Theorem 2. Moreover, we can assume that each non-trivial interval of each of the chains $(\bar{S}, \leqq)$ and, consequently, also of $(S, \leqq)$, contains a copy of the ordinal κ. As a consequence of this remark, we obtain the following positive answer to a question of van Benthem [19; p. 45]. Under the generalized continuum hypotheses, for regular cardinals κ this result has already been established by McCleary [14; Theorem 15]. For $\kappa = \aleph_1$, it is contained in Burgess [22].

Corollary 3.6. *For each uncountable cardinal κ there exists an infinite chain $(S, \leqq)$ of cardinality κ such that each open interval (a, b) of S $(a, b \in S, a < b)$ is isomorphic to $(S, \leqq)$, but $(S, \leqq)$ is not antiisomorphic to itself.*

Proof. By Theorem 2 and the above remarks, there exists a doubly homogeneous chain $(S', \leqq)$ of cardinality κ such that each point $s \in S'$ has cofinality $\aleph_1$ and coinitiality $\aleph_0$, and that each non-trivial interval of S' contains a copy of κ. Choose $a, b \in S'$ with $a < b$ and put $S = (a, b)$. The result follows.

§4. Embedding results for lattice-ordered groups

In this section we wish to prove Theorem 3 and derive from it universal embedding results like Corollary 2. We will also consider embeddings of arbitrary l-groups into l-groups all of whose group automorphisms are inner. As in the previous section, we will frequently obtain collections of doubly homogeneous chains $(S_i, \leqq)$ with pairwise non-isomorphic and non-antiisomorphic Dedekind-completions. We just mention here that then an obvious consequence for the corresponding automorphism groups $A(S_i)$ is immediate by the following result. Let e always denote the identity element in a given group.

PROPOSITION 4.1 (Holland [10], Jambu-Giraudet [11], McCleary [15], Rabinovich [16], cf. [7; §2]). *Let S, T be two doubly homogeneous chains whose Dedekind-completions $\bar{S}$, $\bar{T}$ are neither isomorphic nor antiisomorphic. Then the automorphism groups $A(S)$ and $A(T)$ are non-isomorphic both as groups and as lattices with e.*

Now we turn to the proof of Theorem 3. We will need the following lemma.

LEMMA 4.2. *Let $(S, \leqq)$ be a doubly homogeneous chain and $a_i, b_i \in \bar{S} \setminus S$, $s_i \in S$ such that $a_i < s_i < b_i$ and $\mathrm{coi}(a_i) = \mathrm{cof}(b_i) = \aleph_0$ for $i = 1, 2$. Then there is an isomorphism π from $(a_1, b_1)_S$ onto $(a_2, b_2)_S$ mapping s_1 onto s_2.*

PROOF. For $i = 1, 2$, choose a countable subset $C_i = \{c_{i,j}\,;\, j \in \mathbf{Z}\} \subseteq (a_i, b_i)_S$ which is unbounded above and below in $(a_i, b_i)_S$ such that $c_{i,j} < c_{i,j+1}$ for each $j \in \mathbf{Z}$ and $c_{i,0} = s_i$. For each $j \in \mathbf{Z}$, choose an isomorphism π_j from $[c_{1,j}, c_{1,j+1}]_S$ onto $[c_{2,j}, c_{2,j+1}]_S$, and put $\pi = \bigcup_{j \in \mathbf{Z}} \pi_j$ to obtain the result.

Recall that we identify each automorphism of a chain $(S, \leqq)$ with its (unique) extension to $(\bar{S}, \leqq)$. Now assume that S and $\bar{S} \setminus S$ are dense in $(\bar{S}, \leqq)$. Then each automorphism of $(S, \leqq)$ maps $\bar{S} \setminus S$ onto itself; hence we can identify $A(S)$ with $A(\bar{S} \setminus S)$, in a natural way. This is done in the following result. Its proof uses ideas already contained in Holland [9; proof of Theorem 4].

THEOREM 4.3. *Let $(C, \leqq)$ be a chain, $(S, \leqq)$ a doubly homogeneous chain, and $\varphi: (\bar{C}, \leqq) \to (\bar{S}, \leqq)$ a complete embedding such that $\varphi(\bar{C}) \subseteq \bar{S} \setminus S$. Assume that whenever $a, b \in C$ satisfy $a < b$ and there is no $c \in C$ with $a < c < b$, then the interval $(\varphi(a), \varphi(b))$ in $\bar{S}$ has countable coterminality. Then there exists an l-embedding ψ of $A(\bar{C})$ into $A(S)$ such that (ψ, φ) l-embeds $(A(\bar{C}), \bar{C})$ into $(A(S), \bar{S} \setminus S)$ and ψ maps $B(\bar{C})$ into $B(S)$.*

PROOF. Let P be the set of all pairs $(a \mid b) \in C \times C$ such that $a < b$ and there is no $c \in C$ with $a < c < b$. For each pair $(a \mid b) \in P$ let $D_{(a|b)} = (\varphi(a), \varphi(b))_S$. By assumption and Lemma 4.1, the sets $D_{(a|b)}$ $((a \mid b) \in P)$ are pairwise isomorphic. We now define a family of isomorphisms $\pi_{q,r}: D_q \to D_r$ $(q, r \in P)$ as follows. Fix $p \in P$ and let $\pi_{p,p}$ be the identity mapping on D_p. For each $q \in P$ with $q \neq p$, choose an isomorphism $\pi_{p,q}$ from D_p onto D_q, and let $\pi_{q,p} = \pi_{p,q}^{-1}$. For any $q, r \in P$, let $\pi_{q,r} = \pi_{p,r} \circ \pi_{q,p}$. Then $\pi_{q,t} = \pi_{r,t} \circ \pi_{q,r}$ for any $q, r, t \in P$.

Next let $s \in S$ such that $\varphi(a) < s < \varphi(b)$ for some $a, b \in C$. We claim that $s \in D_q$ for some $q \in P$. Indeed, let $A = \{z \in \bar{C}; \varphi(z) \leqq s\}$ and $B = \{z \in \bar{C}; s \leqq \varphi(z)\}$, and put $c = \sup A$, $d = \inf B$ in $(\bar{C}, \leqq)$. As in the proof of Theorem 1 we obtain $(c \mid d) \in P$ and thus $s \in D_{(c|d)}$.

Now we define a mapping $\psi: A(\bar{C}) \to A(S)$. Let $f \in A(\bar{C})$. We define $f^* \in A(S)$ as follows. For each $q = (a \mid b) \in P$, observe that $r = (f(a) \mid f(b)) \in P$, and put $f^*|_{D_q} = \pi_{q,r}$. For each $s \in S$ with $s < \varphi(C)$ or $\varphi(C) < s$ let $f^*(s) = s$. Then f^* is defined on all of S as shown above, and $f^* \in A(S)$. Now put $\psi(f) = f^*$.

Clearly ψ is a group homomorphism, preserves the lattice operations, and maps $B(\bar{C})$ into $B(S)$. We show that ψ is injective. Let $f \in A(\bar{C})$ such that f^* is the identity on S. Then $f(a) = a$, $f(b) = b$ for each pair $(a \mid b) \in P$. By Theorem 1, for any $c, d \in \bar{C}$ with $c < d$ there is a pair $(a \mid b) \in P$ with $c \leqq a < b \leqq d$. Hence f is the identity on $\bar{C}$. Consequently, ψ is an l-embedding.

To prove that (ψ, φ) is an l-embedding, let $f \in A(\bar{C})$. We claim that $\varphi \circ f = f^* \circ \varphi$ (where f^* is identified with its extension to an automorphism of $\bar{S}$). Clearly, if $c \in \{a, b\}$ for some $(a \mid b) \in P$, we have $(\varphi \circ f)(c) = (f^* \circ \varphi)(c)$. Since again no non-trivial closed interval of C is dense, and φ is a complete embedding, we obtain this identity for all $c \in \bar{C}$. The result follows.

Note that as one of several immediate consequences of Theorem 1 (or Theorem 3.2) and Theorem 4.3 we have

COROLLARY 4.4. *Let $(C, \leqq)$ be an arbitrary chain and $\kappa \geqq |C|$ an uncountable cardinal. Then there exists a doubly homogeneous chain $(S, \leqq)$ of cardinality κ and an l-embedding (ψ, φ) of $(A(C), \bar{C})$ into $(A(S), \bar{S})$ such that φ preserves all suprema of subsets of $\bar{C}$ and ψ maps $B(C)$ into $B(S)$. If no non-trivial closed interval of C is dense, here φ can be chosen to be complete.*

PROOF. The final statement is immediate as indicated above. This implies the first assertion as follows. Let $(C, \leqq)$ be an arbitrary chain, and put $D = C \times \{1, 2\}$, ordered lexicographically. Then, as shown in the proof of Corollary 2.7, there exists a natural l-embedding (ψ, φ) of $(A(C), \bar{C})$ into $(A(D), \bar{D})$ such that φ preserves all suprema of subsets of $\bar{C}$. Now the result follows from the final statement of the corollary.

Next we apply Theorem 3.2 and the proof of Theorem 4.3 to obtain the

PROOF OF THEOREM 3. First let $(C^+, \leqq)$ be the chain C together with a new greatest and a new smallest element. Canonically, $A(C) = B(C^+) = A(C^+)$. Next let $D = C^+ \times \{1, 2\}$ be ordered lexicographically. We identify C^+ with the set $C^+ \times \{1\}$ in the natural way. Clearly $|D| \leqq \kappa$ and no non-trivial closed interval of D is dense. By Theorem 3.2, there are 2^κ doubly homogeneous chains $(S, \leqq)$ of cardinality κ with pairwise non-isomorphic and non-antiisomorphic

Dedekind-completions such that for each of these chains $(S, \leqq)$, there exists a complete embedding φ of $(\bar{D}, \leqq)$ into $(\bar{S}, \leqq)$ and the triple D, S, φ satisfies the assumptions of Theorem 4.3.

We now use the same notation as in the proof of Theorem 4.3 (with C replaced by D) and proceed similarly. For each pair $q = (a \mid b) \in P$ choose an element $s_q \in D_q$, and, by Lemma 4.2, choose the isomorphism $\pi_{p,q}$ such that it maps s_p onto s_q. Then, for any $q, r \in P$, $\pi_{q,r}$ maps s_q onto s_r.

Now we construct the l-embedding ψ: $A(D) \rightarrow A(S)$ as before. Note that each automorphism of C extends uniquely to an automorphism of D, and hence $A(C)$ is an l-subgroup of $A(D)$. Next define an embedding φ^* of $(C, \leqq)$ into $(S, \leqq)$ as follows. If $c \in C$, then $q = ((c,1) \mid (c,2)) \in P$; put $\varphi^*(c) = s_q$. It is straightforward to check that then (ψ, φ^*) is an l-embedding of $(A(C), C)$ into $(B(S), S)$.

Next we obtain the following embedding result for arbitrary l-groups.

Corollary 4.5. *Let G be an arbitrary l-group and $\kappa \geqq |G|$ a regular uncountable cardinal. Then there exist 2^κ doubly homogeneous chains $(S, \leqq)$ of cardinality κ with pairwise non-isomorphic and non-antiisomorphic Dedekind-completions such that G can be l-embedded into $B(S)$.*

Proof. By Holland's theorem [9], there exists a chain $(C, \leqq)$ with cardinality $|C| \leqq |G|$ such that G can be l-embedded into $A(C)$. Now Theorem 3 implies the result.

Let $(S, \leqq)$ be an infinite chain and $G \subseteq A(S)$ a subgroup. Then (G, S) is called a *triply transitive ordered permutation group*, if whenever $A, B \subseteq S$ each have three elements, there exists $g \in G$ with $g(A) = B$. An element $g \in G$ is called *strictly positive*, if $g > e$ in $A(S)$. For the proof of Corollary 2 we will need the following special case of a result of McClearly [15]. It generalizes the group-theoretical part of Proposition 4.1.

Proposition 4.6 (McClearly [15; Main Theorem 4]). *Let (G, S) and (H, T) be two triply transitive ordered permutation groups such that G and H each contain strictly positive elements of bounded support. If G and H are isomorphic (as groups), then $(\bar{S}, \leqq)$ and $(\bar{T}, \leqq)$ are either isomorphic or antiisomorphic.*

Using this result and Corollary 4.5, we now give the

Proof of Corollary 2. By Corollary 4.5, there are 2^κ doubly homogeneous chains $(S_i, \leqq)$ of cardinality κ with pairwise non-isomorphic and non-antiisomorphic Dedekind-completions such that, without loss of generality, G is an l-subgroup of each $B(S_i)$ ($i < 2^\kappa$). For each $i < 2^\kappa$, we proceed as follows. Choose a strictly positive element $z_i \in B(S_i)$ and for each pair (A, B) of three-element-subsets $A, B \subseteq S_i$ an automorphism $f_i(A, B) \in B(S_i)$ which maps A onto B. Let G_i be the l-subgroup of $B(S_i)$ generated by G, z_i, and the automorphisms $f_i(A, B)$. Then $|G_i| = \kappa$. Since $B(S_i)$ is simple and divisible, by an easy induction we now obtain a simple divisible l-subgroup H_i of $B(S_i)$ which contains G_i and has cardinality κ. By Proposition 4.6, the groups H_i ($i < 2^\kappa$) are pairwise non-isomorphic. The result follows.

Again, if $\kappa \geqq |G|$ is a singular cardinal, there are at least $2^{<\kappa}$ pairwise non-isomorphic l-groups H of cardinality κ satisfying the requirements of Corollary 2.

Next we consider embeddings of arbitrary l-groups into l-groups all of whose group automorphisms are inner. It has been shown that if G is an l-group and $\kappa \geqq |G|$ a regular uncountable cardinal, then there exists a doubly homogeneous chain $(S, \leqq)$ of cardinality $|S| = \Sigma_{\lambda<\kappa} 2^\lambda$ such that G can be l-embedded into $A(S)$ and each automorphism of $A(S)$ is inner (McClearly [14, 15], Weinberg [21]). Here we wish to improve this result with regard to the cardinality of the chain S, and to show that there are many non-isomorphic chains S (of a given cardinality) with the above properties. We will need the following result characterizing group-automorphisms and l-group-automorphisms ($= l$-automorphisms) of $A(S)$ (S doubly homogeneous).

Proposition 4.7 (Lloyd [12], Holland [10], McClearly [13], cf. [7; Theorem 2.4.1]). *Let $(S, \leqq)$ be a doubly homogeneous chain.*

(a) *If for no $z \in \bar{S} \setminus S$, the $A(S)$-orbit of z in $\bar{S}$ is isomorphic to S, then every l-automorphism of $A(S)$ is inner.*
(b) *If $(S, \leqq)$ is not antiisomorphic to itself, then every automorphism of $A(S)$ is an l-automorphism.*

The following result is well-known (cf., e.g., [7; Lemma 1.10.10]).

Lemma 4.8. *Let $(S, \leqq)$ be a doubly homogeneous chain. If $x, y \in \bar{S} \setminus S$ both have countable coterminality, they belong to the same $A(S)$-orbit in $\bar{S}$.*

Now we can show:

Theorem 4.9. *Let G be an arbitrary l-group and $\kappa \geqq |G|$ any uncountable cardinal. Then there exists a doubly homogeneous chain $(S, \leqq)$ of cardinality κ*

such that G can be l-embedded into $A(S)$ and every automorphism of $A(S)$ is inner.

PROOF. By Holland's theorem [9], we can assume that $G \subseteq A(C_1)$ for some chain C_1 with $|C_1| \leqq |G|$. Let $D = C_1 \times \{1,2\}$ be ordered lexicographically. Next let T be a copy of ω_1, the first uncountable ordinal, and let $(C, \leqq)$ be the ordinal sum of $(D, \leqq)$ and $(T, \leqq)$, i.e. $C = D \dot{\cup} T$ and $D < T$ in $(C, \leqq)$. By Theorem 3.2 and the remark following it, there exists a doubly homogeneous chain $(S, \leqq)$ of cardinality κ and a complete embedding φ of $(\bar{C}, \leqq)$ into $(\bar{S}, \leqq)$ such that each non-trivial interval of $\bar{S}$ contains a copy of κ, and conditions (i)–(iii) of Theorem 3.2 are satisfied. We may assume that $\varphi(T)$ is unbounded above in S. Indeed, otherwise let $s = \sup \varphi(C)$ and consider $\{x \in S; x < s\}$ instead of S. Consequently, S has cofinality $\aleph_1$. Since each non-trivial interval of $\bar{S}$ contains a copy of κ, this procedure does not change the cardinality of S. Similarly, we may assume that S has countable coinitiality. Hence $\bar{S}$ is not antiisomorphic to itself.

Clearly, $A(C_1)$ can be l-embedded into $A(D)$, $A(D)$ into $A(C)$, and, by Theorem 4.3, $A(C)$ into $A(S)$. Hence it only remains to show that each automorphism of $A(S)$ is inner. By Proposition 4.7, it suffices to show that there is no $A(S)$-orbit Z in $\bar{S} \setminus S$ which is isomorphic to S. Suppose there were such an $A(S)$-orbit Z in $\bar{S} \setminus S$ and an isomorphism $\psi: Z \to S$. Since each $s \in S$ has countable coterminality, the same applies to all $z \in Z$. By Lemma 4.8, Z contains precisely all points $x \in \bar{S} \setminus S$ which have countable coterminality. Let ψ also denote the extension of ψ to an automorphism of $\bar{S}$. The set $A = \varphi(T)$, and hence also $B = \psi(A)$, is closed and unbounded above in $\bar{S}$. By Lemma 3.1, there exists a club $E \subseteq A \cap B$ on which ψ acts like the identity. Thus $E \cap Z = \varnothing$. However, A is isomorphic to ω_1, and our assumption on φ implies that each element of A (except maybe the first) has countable coterminality, showing $E \subseteq A \subseteq Z \cup \{\min A\}$, a contradiction. The result follows.

We remark that it is not difficult to combine the ideas of the proofs of Theorems 3.2 and 4.9 in order to sharpen the existence result of Theorem 4.9. We only have to split the (stationary) subset of all countable limit-elements of T into two disjoint stationary subsets T_1 and T_2, and perform by Solovay's result the splitting-argument of the proof of Theorem 3.2 only on the set T_2 (instead of T). Then each element of $\varphi(T_1)$ has countable coterminality in $\bar{S}$, and in the above proof of Theorem 4.9 we obtain the final contradiction by $E \cap \varphi(T_1) \neq \varnothing$. Hence we have shown:

COROLLARY 4.10. *Let G be an arbitrary l-group and $\kappa \geqq |G|$ an uncountable cardinal. If κ is regular, there exist 2^κ doubly homogeneous chains $(S, \leqq)$ of* *cardinality κ with pairwise non-isomorphic and non-antiisomorphic Dedekind-completions such that G can be l-embedded into $A(S)$ and every automorphism of $A(S)$ is inner. If κ is singular, there are at least $2^{<\kappa} = \Sigma_{\lambda<\kappa} 2^{\lambda}$ such chains $(S, \leqq)$.*

As pointed out earlier, all these automorphism groups $A(S)$ are non-isomorphic both as groups and as lattices with e. Indeed, many of the doubly homogeneous chains $(S, \leqq)$ can be constructed in a way such that the automorphism groups $A(S)$ are even elementarily inequivalent as groups (or, equivalently, as lattices with e; cf. [7, 11]). We put this remark into a more general context. For any group G, let (g) denote the smallest normal subgroup of G containing $g \in G$; we put $N_1(G) = \{(g); g \in G\}$. In [1, 3, 4, 6–9] the structure of the normal subgroup lattice of the groups $A(S)$ (S a doubly homogeneous chain) was studied; it is determined uniquely by the structure of its partially ordered subset $(N_1(A(S)), \subseteq)$. The following result is essentially contained in Ball and Droste [1].

PROPOSITION 4.11. *Let S, T be two doubly homogeneous chains. If $A(S)$ and $A(T)$ are elementarily equivalent as groups, then $(N_1(A(S)), \subseteq)$ and $(N_1(A(T)), \subseteq)$ are elementarily equivalent as partially ordered sets.*

PROOF. As shown in [1; Theorem 3.12], there is a single formula $\varphi(x, y)$ in the first order language of predicate calculus for group theory such that whenever $f, g \in A(S)$, then $\varphi[f, g]$ holds in $A(S)$ iff $(f) \subseteq (g)$. This implies the result.

Now the results of [3; (7.6)–(7.8)] show that for many of the doubly homogeneous chains $(S, \leqq)$ constructed in the proof of Theorem 3.2, the partially ordered sets $(N_1(A(S)), \subseteq)$ are elementarily inequivalent, hence so are the automorphism groups $A(S)$ by Proposition 4.11. These remarks will be further utilized in a later paper (see M. Droste, *Normal subgroups and elementary theories of lattice-ordered groups*, to appear).

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